CHAPTER 3

FRACTIONAL CALCULUS

[*This chapter is based on the lectures of Dr. R.K. Saxena of the jai Narain Vyas University, Jodhpur, Rajasthan, India at the 2008 SERC School.*]

3.0. Introduction

Definition 3.0.1.

The subject of fractional calculus deals with the investigation of integrals and derivatives of arbitrary real or complex order. The concept of fractional calculus appears to have stemmed from a question raised by Marquis de l′Hopital in the year 1695 to Gottfried Wilhelm Leibniz regarding the meaning of Leibniz notation

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n},\ n \in N_0 = 0.1, 2, ...$$

the derivative of order n, when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). Leibniz replied to l′Hopital on 30 September 1695 as follows: “ This is an apparent paradox from which one day useful consequences will be drawn”.

The importance of this subject lies in the fact that during the last four decades, three international conferences dedicated exclusively to fractional calculus and its applications were held in the University of New Haven in 1974, University of Strathclyde, Glasgow, Scotland in 1984, and the third in Nihon University in Tokyo, Japan in 1989 in which various workers presented their investigations dealing with the theory and applications of fractional calculus. (see, for details, Ross (1975), McBride and Roach (1985) and Nishimoto (1990)). Fractional calculus is applicable in many areas of science and engineering, such as fluid flow, rheology, electric network, viscoelasticity, electrochemistry of corrosion etc.

In this section, we will present a brief introduction and theory of Riemann-Liouville, Weyl, Erdélyi-Kober and Saigo operators of fractional calculus. A detailed and comprehensive account of various fractional calculus operators is available from the books by Oldham and Spanier (1974), Miller and Ross(1993), Nishimoto (1991), Podlubny (1999), Samko et al (1993), Hilfer (2000), Kilbas et al(2006), and a paper written by Srivastava and Saxena (2001).

3.1. Fractional Calculus Operators of Arbitrary Order

In this section, we will discuss the definitions and basic properties of the Riemann-Liouville, Weyl and Erdélyi-Kober operators of fractional calculus

3.1.1. Riemann-Liouville fractional integrals of order α

Notation 3.1.1. ${}_aI_x^\alpha, {}_aD_x{}^{-\alpha}; I_{a+}^\alpha$: Riemann-Liouville left-sided fractional integral of order α.

Notation 3.1.2. ${}_xI_b^\alpha, {}_xD_b^{-\alpha}; I_{b-}^\alpha$: Riemann-Liouville right-sided fractional integral of order α.

Definition 3.1.1. Let $f(x) \in L(a,b),\ \alpha \in C,\ \Re(\alpha) > 0$, then

$$ {}_aI_x^\alpha f(x) = {}_aD_x^{-\alpha} f(x) = I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t,\ x > a \tag{3.1.1} $$

is called the Riemann-Liouville left-sided fractional integral of order α.

Definition 3.1.2. Let $f(x) \in L(a,b),\ \alpha \in C,\ \Re(\alpha) > 0$, then

$$ {}_xI_b^\alpha f(x) = {}_xD_b^{-\alpha} f(x) = I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)\mathrm{d}t,\ x < b \tag{3.1.2} $$

is called the Riemann-Liouville right-sided fractional integral of order α.

3.1.2. Basic properties of fractional integrals

Property 3.1.1. Fractional integrals obey the following property:

$$ {}_aI_x^\alpha {}_aI_x^\beta \varphi = {}_aI_x^{\alpha+\beta} \varphi = {}_aI_x^\beta {}_aI_x^\alpha \varphi. $$

$$ {}_xI_b^\alpha {}_xI_b^\beta \varphi = {}_aI_x^{\alpha+\beta} \varphi = {}_xI_b^\beta {}_xI_b^\alpha \varphi. \tag{3.1.3} $$

Proof 3.1.1. By virtue of the definition (3.1.1), and well-known Dirichlet formula, namely

$$ \int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\mathrm{d}x \tag{3.1.4} $$

it follows that

$$ \begin{aligned} {}_aI_x^\alpha {}_aI_x^\beta \varphi &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}} \frac{1}{\Gamma(\beta)} \int_a^t \frac{\varphi(u)\mathrm{d}u}{(t-u)^{1-\beta}} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \mathrm{d}u\varphi(u) \int_u^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}(t-u)^{1-\beta}}, \end{aligned} \tag{3.1.5} $$

If we use the substitution $y = \frac{t-u}{x-u}$, the value of the second integral is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)(x-u)^{1-\alpha-\beta}}\int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y = \frac{(x-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

which when substituted in (3.1.5) yields the first part of (3.1.3). The second part can be similarly established. In particular,

$$_aI_x^{n+\alpha} f = {}_aI_x{}^n{}_aI_x^{\alpha} f,\ n \in N,\ \Re(\alpha) > 0 \tag{3.1.6}$$

which shows that the n-fold differentiation

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} I_x^{n+\alpha} f(x) = {}_aI_x^{\alpha} f(x),\ n \in N,\ \Re(\alpha) > 0 \tag{3.1.7}$$

for all x. When $\alpha = 0$, we obtain

$$_aI_x^0 f(x) = f(x);\ {}_aI_x^{-n} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = f^{(n)}(x). \tag{3.1.8}$$

Note 3.1.1. The property given in (3.1.3) is called the semigroup property of fractional integration.

Notation 3.1.3. $L(a,b)$, Space of Lebesgue measurable real or complex valued functions.

Definition 3.1.3. $L(a,b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a,b]$:

$$L(a,b) = \{f : \|f\|_1 = \int_a^b |f(t)|\mathrm{d}t < +\infty\}. \tag{3.1.9}$$

Note 3.1.2. The operators $_aI_x^{\alpha}$ and $_xI_b^{\alpha}$ are defined on the space $L(a,b)$.

Property 3.1.2. The following result holds:

$$\int_a^b f(x)({}_aI_x^{\alpha} g)\mathrm{d}x = \int_a^b g(x)({}_xI_b^{\alpha} f)\mathrm{d}x. \tag{3.1.10}$$

The result (3.1.10) can be established by interchanging the order of integration in the integral on the left of (3.1.10) and then using the Dirichlet formula (3.1.4).

Example 3.1.1. If $f(x) = (x-a)^{\beta-1}$, then find the value of $_aI_x^{\alpha} f(x)$.

Solution 3.1.1. We have

$$_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\mathrm{d}t.$$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},$$

where $\Re(\beta) > 0$. Thus

$$_aI_x^{\alpha} f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}. \tag{3.1.11}$$

It can be similarly shown that

$${}_xI_b^\alpha g(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}, \ x < b, \tag{3.1.12}$$

where $g(x) = (b-x)^{\beta-1}$.

Note 3.1.3. It may be noted that (3.1.11) and (3.1.12) give the Riemann- Liouville integrals of the power functions $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$, $\Re(\beta) > 0$.

Exercises 3.1

3.1.1. Prove that

$$\left({}_aI_x^\alpha (x \pm c)^{\gamma-1}\right)(x) = \frac{(a \pm c)^{\gamma-1}}{\Gamma(\alpha+1)}(x-a)^\alpha {}_2F_1(1, 1-\gamma; \alpha\frac{a-x}{a \pm c}),$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \gamma \in C, a < x < b$.

3.1.2. Prove that

$$\left({}_aI_x^\alpha (x-a)^{\beta-1}(b-x)^{\gamma-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{1-\gamma}} {}_2F_1(\beta, 1-\gamma; \alpha+\beta; \frac{x-a}{b-a}),$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \gamma \in C, a < x < b$.

3.1.3. Prove that

$$\left(I_x^\alpha [\mathrm{e}^{\lambda x}]\right)(x) = \mathrm{e}^{\lambda a}(x-a)^\alpha E_{1,\alpha+1}(\lambda x - \lambda a).$$

where $x > a$, $\Re(\alpha) > 0$, and $E_{1,\alpha+1}(\cdot)$ is the Mittag-Leffler function, defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta \in C; \Re(\alpha) > 0, \Re(\beta) > 0.$$

3.1.4. Prove that

$$\left(I_x^\alpha [\mathrm{e}^{\lambda x}(x-a)^{\beta-1}]\right)(x) = \mathrm{e}^{\lambda a}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1} {}_1F_1(\beta; \alpha+\beta; \lambda x - \lambda a)$$

where $\min\{\Re(\alpha), \Re(\beta)\} > 0$,

3.1.5. Prove that

$$\left({}_aI_x^\alpha (x-a)^{\beta-1}\ln(x-a)\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}[\ln(x-a) + \psi(\beta) - \psi(\alpha+\beta)],$$

where $\min\{\Re(\alpha), \Re(\beta)\} > 0$; and $\psi(\cdot)$ is the logarithmic derivative of the gamma function.

3.1.6. Prove that

$$\left({}_aI_x^{\alpha}(x-a)^{\nu/2}J_\nu[\lambda\sqrt{x-a}]\right)(x)=\left(\frac{2}{\lambda}\right)^{\nu}(x-a)^{(\alpha+\nu)/2}J_{\alpha+\nu}(\lambda\sqrt{x-a}),$$

where $\Re(\alpha)>0,\Re(\nu)>-1$.

3.1.7. Prove that

$$\left({}_aI_x^{\nu}(x-a)^{\beta-1}E_{\mu,\beta}[(x-a)]^{\mu}\right)(x)=(x-a)^{\nu+\beta-1}E_{\mu,\nu+\beta}[(x-a)^{\mu}],$$

where $\Re(\nu)>0$.

3.1.8. Prove that

$$\begin{aligned}\left({}_aI_x^{\nu}x^{\mu-1}\sin ax\right)(x) &= \frac{x^{\mu+\nu-1}}{2i}\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}\\ &\quad\times\left[{}_1F_1(\mu;\mu+\nu;iax)-[{}_1F_1(\mu;\mu+\nu;-iax)]\right],\end{aligned}$$

where $a>0,\min\{\Re(\nu),\Re(\mu)\}>0$.

3.1.9. Prove that Riemann-Liouville fractional integral is bounded. That is

$$\|{}_aI_x^{\alpha}h\|_1\le\frac{(b-a)^{\Re(\alpha)}}{|\Re(\alpha)|\Re(\alpha)}\|h\|_1,$$

where $\alpha\in C,\Re(\alpha)>0$.

3.1.3. Derivatives of fractional order

In this section, we study various fractional order derivatives which occur in certain reaction (relaxation) and diffusion problems.

Notation 3.1.4. $\{\alpha\}$ means the fractional part of number α, $0\le\{\alpha\}<1$.

Notation 3.1.5. $[\alpha]$ means the integral part of number α.

Note 3.1.4. We note that

$$\alpha=\{\alpha\}+[\alpha].$$

Notation 3.1.6. ${}_aD_x^{\alpha}\varphi(x);D_{a+}^{\alpha}\varphi(x)$: Riemann-Liouville left-sided fractional derivative of the function $\varphi(x)$ of order α

Notation 3.1.7. ${}_bD_x^{\alpha}\varphi(x),I_{b-}^{\alpha}\varphi(x)$: Riemann-Liouville right-sided fractional derivative of the function $\varphi(x)$, of order α.

Definition 3.1.4. The left-sided Riemann-Liouville derivative of order $\alpha>0$ of the function $\varphi(x)$ is defined by

$${}_aD_x^{\alpha}\varphi(x)=D_{a+}^{\alpha}\varphi(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x\frac{\varphi(t)dt}{(t-x)^{\alpha-n+1}},\ n=[\alpha]+1 \tag{3.1.13}$$

Definition 3.1.5. The right-sided Riemann-Liouville derivative of order $\alpha > 0$ of the function $\varphi(x)$ is defined by

$$ {}_xD_b^\alpha\varphi(x) = D_{b-}^\alpha\varphi(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{\varphi(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}},\; n = [\alpha]+1 \qquad (3.1.14)$$

In short, one can express (3.1.13) in the form

$$ {}_aD_x^\alpha\varphi(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}_aI_x^{n-\alpha}\varphi(x) \qquad (3.1.15)$$

and (3.1.14) as

$$ {}_xD_b^\alpha\varphi(x) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}_xI_b^{n-\alpha}\varphi(x). \qquad (3.1.16)$$

We shall also employ the notations

$$ {}_aD_x^\alpha\varphi =_a I_x^{-\alpha}\varphi = ({}_aI_x^\alpha)^{-1}\varphi;\; \alpha \geq 0, \qquad (3.1.17)$$

Similarly, we have

$$ {}_xD_b^\alpha\varphi = {}_xI_b^{-\alpha}\varphi = ({}_xI_b^\alpha)^{-1}\varphi;\; \alpha \geq 0, \qquad (3.1.18)$$

Example 3.1.2. Prove that

$$ {}_0D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}, \alpha \geq 0, \gamma > -1,\; x > 0 \qquad (3.1.19)$$

Solution 3.1.2. We have

$$\begin{aligned} {}_0D_x^\alpha(x^\gamma) &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x t^\gamma(x-t)^{n-\alpha-1}\mathrm{d}t \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+n+1-\alpha)}(\gamma-\alpha+1)_n x^{\gamma-\alpha} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma-\alpha}, \end{aligned}$$

for $\gamma > -1,\; (\gamma-\alpha+1)_n \neq 0$.

Note 3.1.5. It is interesting to observe that for $\gamma = 0$, (3.1.19) reduces to

$$ D_x^\alpha 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}; \alpha \neq 1, 2, \ldots. \qquad (3.1.20)$$

which is an interesting result and indicates that the fractional derivative of a constant is not zero.

Remark 3.1.1. Podlubny (2002) has given the geometric and physical interpretation of the fractional integration and fractional differentiation.

Exercises 3.1

3.1.10. Prove that

$$ \left({}_0I_x^\alpha[x^p\exp(ax)]\right)(x) = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}\,{}_1F_1(p+1; p-\alpha+1; ax)$$

where $p \neq -1, -2, -3, \cdots$.

3.1.11. Prove that

$$J_\nu(x) = \pi^{-1/2} 2^{1-\nu} x^{-\nu} {}_0D_x{}^{-\nu+1/2}(\sin x).$$

3.1.12. Prove that

$$\psi(x) = -\gamma + \ln z - \Gamma(x) z^{1-x} {}_0D_z^{1-x}(\ln z),$$

3.1.13. Prove that

$$\gamma(a, z) = \Gamma(a) \mathrm{e}^{-z} {}_0D_z^{-a}(\exp z),$$

where $\gamma(a, z)$ is the incomplete gamma function.

3.1.14. Prove that

$$\left({}_0D_x^\nu x^{\frac{\mu}{2}} J_\mu x^{\frac{1}{2}} \right)(x) = 2^{-\nu} x^{\frac{1}{2}(\mu-\nu)} J_{\mu-\nu}(x^{\frac{1}{2}}).$$

where $\Re(\mu) > -1$.

3.1.15. Prove that

$${}_0I_x^\nu \ln x = \frac{x^\nu}{\Gamma(\nu + 1)}[\ln x - \gamma - \psi(\nu + 1)],$$

3.1.16. Prove that

$${}_0D_x^\alpha (x + a)^p = \frac{a^p x^{-\alpha}}{\Gamma(1 - \alpha)} {}_2F_1(1, -p; 1 - \alpha; -x/a).$$

3.1.17. Prove that

$$({}_aD_x^\alpha {}_aI_x^\alpha h)(x) = h(x),$$

holds almost everywhere on [a,b], where $h(x) \in L(a, b),\ \alpha \in C,\ \Re(\alpha) > 0$.

3.1.18. Prove that

$$({}_aD_x^\beta {}_aI_x^\alpha h)(x) = {}_aI_x^{\alpha-\beta} h(x),$$

where $\alpha,\ \beta \in C,\ \min\{\Re(\alpha),\ \Re(\beta)\} > 0; h(x) \in L(a, b)$.

3.1.4. The Weyl integral

Notation 3.1.8. ${}_xW_\infty^\alpha,\ {}_xI_\infty^\alpha$. Weyl integral of order α.

Definition 3.1.6. The Weyl integral of $f(x)$ of order α, denoted by ${}_xW_\infty^\alpha$, is defined by

$${}_xW_\infty^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t - x)^{\alpha-1} f(t)\mathrm{d}t,\ -\infty < x < \infty \tag{3.1.21}$$

where $\alpha \in C,\ \Re(\alpha) > 0$. (3.1.21) is also denoted by $I_-^\alpha f(x)$.

Notation 3.1.9. ${}_xD^\alpha_\infty$, D^α_-: Weyl fractional derivative.

Definition 3.1.7. The Weyl integral of $f(x)$ of order α, denoted by ${}_xD^\alpha_\infty$, is defined by

$$\begin{aligned} {}_xD^\alpha_\infty f(x) = D^\alpha_- f(x) &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left({}_xW^{m-\alpha}_\infty f(x)\right) \\ &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_x^\infty \frac{f(t)\mathrm{d}t}{(t-x)^{1+\alpha-m}}, \; -\infty < x < \infty. \quad (3.1.22) \end{aligned}$$

3.1.5. Basic properties of Weyl integrals.

Property 3.1.3. The following result holds.

$$\int_0^\infty \varphi(x) \left({}_0I^\alpha_x \psi(x)\right) \mathrm{d}x = \int_0^\infty \psi(x) \left({}_0W^\alpha_x \varphi(x)\right) \mathrm{d}x. \quad (3.1.23)$$

(3.1.23) is called the formula for fractional integration by parts. It is also called the Parseval equality. (3.1.23) can be established by interchanging the order of integration.

Property 3.1.4. Weyl fractional integrals obey the semigroup property. That is

$$\left({}_xW^\alpha_\infty \, {}_xW^\beta_\infty f\right) = {}_xW^{\alpha+\beta}_\infty f = \left({}_xW^\beta_\infty {}_xW^\alpha_\infty f\right). \quad (3.1.24)$$

Proof 3.1.2. We have

$$\left({}_xW^\alpha_\infty {}_xW^\beta_\infty f\right) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \mathrm{d}t(t-x)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_t^\infty (u-t)^{\alpha-1} f(u)\mathrm{d}u.$$

Using the modified form of Dirichlet formula (3.1.4) namely

$$\int_x^a \mathrm{d}t(t-x)^{\alpha-1} \int_t^a (u-t)^{\beta-1} f(u)\mathrm{d}u = B(\alpha,\beta) \int_t^a (u-t)^{\alpha+\beta-1} f(u)\mathrm{d}u, \quad (3.1.25)$$

and letting $a \to \infty$, (3.1.25) yields the desired result

$$\left({}_xW^\alpha_\infty {}_xW^\beta_\infty f\right) = {}_xW^{\alpha+\beta}_\infty f.$$

Example 3.1.3. Prove that

$${}_xW^\alpha_\infty \mathrm{e}^{-\lambda x} = \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha},$$

where $\Re(\alpha) > 0$.

Solution 3.1.3. We have

$$\begin{aligned} {}_xW^\alpha_\infty \mathrm{e}^{-\lambda x} &= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \mathrm{e}^{-\lambda t}\mathrm{d}t, \; \lambda > 0, \\ &= \frac{\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} \mathrm{e}^{-\lambda u}\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha}, \; \Re(\alpha) > 0. \end{aligned}$$

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Example 3.1.4. Find the value of ${}_xD_\infty^\alpha e^{-\lambda x},\ \lambda > 0.$

Solution 3.1.4. We have

$$\begin{aligned} {}_xD_\infty^\alpha e^{-\lambda x} &= (-1)^m \left(\frac{d}{dx}\right)^m {}_xW_\infty^{m-\alpha} e^{-\lambda x} \\ &= (-1)^m \left(\frac{d}{dx}\right)^m \lambda^{-(m-\alpha)}\, e^{-\lambda x} \\ &= \lambda^\alpha e^{-\lambda} \end{aligned}$$

Exercises 3.1

3.1.19. Prove that

$$\left({}_xW_\infty^\nu[x^{-\frac{\mu}{2}} J_\mu(ax^{\frac{1}{2}})]\right)(x) = 2^\nu a^{-\nu} x^{\frac{(\nu-\mu)}{2}} J_{\mu-\nu}(ax^{\frac{1}{2}}),$$

where $a,\ x > 0,\ 0 < \Re(\nu) < \frac{3}{4} + \frac{1}{2}\Re(\mu)$.

3.1.20. Prove that

$$({}_aW_\infty^\nu[x^{-\lambda}(x+a)^\mu])(x) = \frac{x^{\mu+\nu-\lambda}\Gamma(\lambda-\mu-\nu)}{\Gamma(\lambda-\mu)} {}_2F_1(-\mu, \lambda-\mu; \nu; \lambda-\mu; -\frac{a}{x}),$$

where $0 < \Re(\nu) < \Re(\lambda-\mu)$; $|\arg(\frac{a}{x})| < \pi$ or $|\frac{a}{x}| < 1$, $\Re(\nu) > 0$.

3.1.21. Prove that

$$({}_aW_\infty^\nu[x^{\nu-1}\exp(-ax)])(x) = \pi^{-\frac{1}{2}}(\frac{x}{a})^{\frac{\nu-1}{2}} \exp(-\frac{ax}{2}) K_{\frac{\nu-1}{2}}(\frac{ax}{2}),$$

where $\Re(ax) > 0$, $\Re(\nu) > 0$.

3.1.22. Prove that

$$({}_xW_\infty^\nu[x^{-2\nu}\exp(\frac{a}{x})])(x) = (\frac{\pi}{x})^{\frac{1}{2}} a^{-\frac{\nu+1}{2}} \exp(\frac{a}{2x}) I_{\frac{\nu-1}{2}}(\frac{a}{2x}),$$

where $\Re(\nu) > 0$.

3.1.6. Erdélyi-Kober operators

Erdélyi-Kober operators are the generalization of Riemann-Liouville and Weyl operators. These operators have been used by many authors in deriving the solution of single, dual and triple integral equations possessing special functions of mathematical physics, as their kernels.

Notation 3.1.10. $I(\alpha, \eta; f(x)),\ E_{0,x}^{\alpha,\eta} f,\ I_x^{\eta,\alpha} f,\ (I_{\eta,\alpha}^+ f)(x)$:Erdélyi- Kober fractional integral of the first kind

Notation 3.1.11. $R[f(x)],\ R(\alpha, \eta; f(x)),\ E_{x,\infty}^{\alpha,\eta} f,\ K_x^{\eta,\alpha} f,\ (K_{\eta,\alpha}^- f)(x)$: Erdélyi-Kober fractional integral of the second kind.

Definition 3.1.8.

$$\begin{aligned} I[f(x)] &= I(\alpha,\eta;f(x))E_{0,x}^{\alpha,\eta}f = I_x^{\eta,\alpha}f = (I_{\eta,\alpha}^{+}f)(x) \\ &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x t^{\eta}(x-t)^{\alpha-1}f(t)\mathrm{d}t,\ \alpha,\eta\in C;\ \Re(\alpha)>0 \end{aligned} \quad (3.1.26)$$

Definition 3.1.9.

$$\begin{aligned} R[f(x)] &= R(\alpha,\zeta;f(x)) = K_{x,\infty}^{\alpha,\zeta}f = K_x^{\zeta,\alpha}f = (K_{\zeta,\alpha}^{-}f)(x) \\ &= \frac{x^{\zeta}}{\Gamma(\alpha)}\int_x^{\infty} t^{-\zeta-\alpha}(t-x)^{\alpha-1}f(t)\mathrm{d}t,\ \alpha,\zeta\in C;\ \Re(\alpha)>0 \end{aligned} \quad (3.1.27)$$

(3.1.26) and (3.1.27) exist under the following set of conditions :

$$f\in L_p(0,\infty),\ \Re(\alpha)>0,\ \Re(\eta)>-\frac{1}{q},\ \Re(\zeta)>-\frac{1}{p},\frac{1}{p}+\frac{1}{q}=1,\ p\geq 1.$$

When $\eta = 0$, (3.1.26) reduces to Riemann-Liouville operator. That is

$$I_x^{0,\alpha}f = x^{-\alpha}\,{}_0I_x^{\alpha}f. \quad (3.1.28)$$

For $\zeta = 0$, (3.1.27) yields the Weyl operator of the function $t^{-\alpha}f(t)$. That is

$$K_x^{0,\alpha}f = {}_xW_{\infty}^{\alpha}t^{-\alpha}f(t). \quad (3.1.29)$$

Theorem 3.1.1. *(Kober, 1940)*

If $\Re(\alpha)>0,\ \Re(\eta-s)>-1,\ f\in L_p(0,\infty),\ 1\leq p\leq 2$ (or $f\in M_p(0,\infty)$), a subspace of $L_p(0,\infty)$ and $p>2$), $\Re(\eta)>-1/q;\ \frac{1}{p}+\frac{1}{q}=1$, then there holds the formula

$$M\{I(\alpha,\eta)f\}(s) = \frac{\Gamma(1+\eta-s)}{\Gamma(\alpha+\eta+1-s)}M\{f(x);s\}. \quad (3.1.30)$$

Proof: Exercise

Theorem 3.1.2. *(Kober, 1940)*

If $\Re(\alpha)>0,\ \Re(\zeta+s)>0,\ f\in L_p(0,\infty),\ 1\leq p\leq 2$ (or $f\in M_p(0,\infty)$), a subspace of $L_p(0,\infty)$ and $p>2$), $\Re(\zeta)>-1/p;\ \frac{1}{p}+\frac{1}{q}=1$, then there holds the formula

$$M\{R(\alpha,\zeta)f\}(s) = \frac{\Gamma(\zeta+s)}{\Gamma(\alpha+\zeta+s)}M\{f(x);s\}. \quad (3.1.31)$$

Proof: Exercise.

Semigroup property of the Erdélyi-Kober operators has been given in the form of the following theorem, which can be proved in the same way:

Theorem 3.1.3. *If* $\Re(\alpha) > 0$, $\Re(\eta) > \max\{-1/p, -1/q\}$; $f \in L_p(0,\infty)$, $g \in L_q(0,\infty)$, $1 \leq p \leq 2$ *(or* $f \in M_p(0,\infty)$*, a subspace of* $L_p(0,\infty)$ *and* $p > 2$*),* $\frac{1}{p} + \frac{1}{q} = 1$*, then there holds the formula*

$$\int_0^\infty g(x)\,(I(\alpha,\eta;f))\,(x)\mathrm{d}x = \int_0^\infty f(x)\,(R(\alpha,\eta;g))\,(x)\mathrm{d}x. \tag{3.1.32}$$

Remark 3.1.2. Operators more general than the operators defined by (3.1.26) and (3.1.27) are defined by Galué et al (2000) in the form

$$(I_{0+}^{\alpha,0,\eta}f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_a^x t^\eta(x-t)^{\alpha-1}f(t)\mathrm{d}t,\ \alpha,\eta \in C;\ \Re(\alpha) > 0. \tag{3.1.33}$$

Exercises 3.1

3.1.23. For the Erdélyi-Kober operators defined by

$$I_{\eta,\alpha}^{+}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-t^2)^{\alpha-1}t^{2\eta+1}f(t)\mathrm{d}t,$$

where $\Re(\alpha) > 0$, establish the following results (Sneddon, 1975):

(i) $I_{\eta,\alpha}^{+}x^{2\beta}f(x) = x^{2\beta}I_{\eta+\beta,\alpha}^{+}f(x)$.

(ii) $I_{\eta,\alpha}^{+}I_{\eta+\alpha,\beta}^{+} = I_{\eta,\alpha+\beta}^{+} = I_{\eta+\alpha,\beta}^{+}I_{\eta,\alpha}^{+}$.

(iii) $(I_{\eta,\alpha}^{+})^{-1} = I_{\eta+\alpha,-\alpha}^{+}$

Remark 3.1.3. The results of Exercise 3.1.23 also hold for the operator, defined by

$$K_{\eta,\alpha}^{-}f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_x^\infty (t^2-x^2)^{\alpha-1}t^{-2\alpha-2\eta+1}f(t)\mathrm{d}t,$$

where $\Re(\alpha) > 0$.

3.1.24. Prove Theorem 3.1.1.

3.1.25. Prove Theorem 3.1.2.

3.2. Generalized Fractional Calculus Operators

3.2.1. Saigo operators

Following Saigo (1978), we define the following generalized fractional calculus operators associated with Gauss hypergeometric function in the kernel and derive their special cases. These operators are useful in the study of certain boundary value problems arising in applied sciences.

Notation 3.2.1. $I_{0+}^{\alpha,\beta,\gamma}$: Left-sided generalized fractional integral operator

Notation 3.2.2. $I_{-}^{\alpha,\beta,\gamma}$: Right-sided generalized fractional integral operator

Notation 3.2.3. $D_{0+}^{\alpha,\beta,\gamma}$: Left-sided generalized fractional derivative operator

Notation 3.2.4. $D_{-}^{\alpha,\beta,\gamma}$: Right-sided generalized fractional derivative operator

Let $\alpha, \beta, \eta \in C$, and let $x \in \Re_+$ the generalized fractional integral and generalized fractional derivative of a function $f(x)$ on $\Re_+$ are defined in the following forms:

Definition 3.2.1.

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t)\mathrm{d}t, \ \Re(\alpha) > 0 \tag{3.2.1}$$

$$= \frac{\mathrm{d}^n}{\mathrm{d}x^n} (I_{0+}^{\alpha+n,\beta-n,\eta-n} f)(x), \ \Re(\alpha) \le 0; \ n = [\Re(-\alpha)] + 1. \tag{3.2.2}$$

Definition 3.2.2.

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}) f(t)\mathrm{d}t, \ \Re(\alpha) > 0 \tag{3.2.3}$$

$$= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} (I_{-}^{\alpha+n,\beta-n,\eta} f)(x), \ \Re(\alpha) \le 0; n = [\Re(-\alpha)] + 1. \tag{3.2.4}$$

Definition 3.2.3.

$$(D_{0+}^{\alpha,\beta,\gamma} f)(x) = (I_{0+}^{-\alpha,-\beta,\alpha+\eta} f)(x)$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f)(x), \ \Re(\alpha) > 0; n = [\Re(\alpha)] + 1 \tag{3.2.5}$$

Definition 3.2.4.

$$(D_{-}^{\alpha,\beta,\gamma} f)(x) = (I_{-}^{-\alpha,-\beta,\alpha+\eta} f)(x)$$

$$= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f)(x), \ \Re(\alpha) > 0; n = [\Re(\alpha)] + 1. \tag{3.2.6}$$

For $\beta = -\alpha$, the operators defined by (3.2.1), (3.2.3), (3.2.5) and (3.2.6) reduce to the following classical Riemann-Liouville fractional calculus operators, see (Samko et al, 1990, Sections 2.3.2.4 and 5.1), defined below by (3.2.7), (3.2.8), (3.2.9) and (3.2.10) respectively.

Definition 3.2.5.

$$(I_{0+}^{\alpha,-\alpha,\eta} f)(x) = {}_0I_x^\alpha f(x) = (I_{0+}^\alpha f)(x) = {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \tag{3.2.7}$$

(Riemann-Liouville)

where $x > 0; \alpha \in C, \Re(\alpha) > 0$.

Definition 3.2.6.

$$(I_{-}^{\alpha,-\alpha,\eta} f)(x) = (I_{-}^{\alpha} f)(x) = {}_{x}W_{\infty}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t)\mathrm{d}t, \tag{3.2.8}$$

(Weyl)

where $x > 0;\ \alpha \in C,\ \Re(\alpha) > 0$.

Definition 3.2.7.

$$\begin{aligned}(D_{+}^{\alpha,-\alpha,\eta} f)(x) &= (D_{0+}^{\alpha} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])} \int_{0}^{x} \frac{f(t)}{(x-t)^{\alpha-[\Re(\alpha)]}} \mathrm{d}t \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \left(I_{0+}^{1-\{\alpha\}} f\right)(x),\ \alpha > 0. \end{aligned} \tag{3.2.9}$$

Definition 3.2.8.

$$\begin{aligned}(D_{-}^{\alpha,-\alpha,\eta} f)(x) &= (D_{-}^{\alpha} f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_{x}^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}} \mathrm{d}t \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \left(I_{-}^{1-\{\alpha\}} f\right)(x)\ \alpha > 0. \end{aligned} \tag{3.2.10}$$

If we set $\beta = 0$, then the operators defined by (3.2.1) and (3.2.3) yield the Erdélyi-Kober operators defined in (3.1.26) and (3.1.27) respectively.

3.2.2. Relation connecting the operators

We note that the relation connecting the operators (3.2.1) and (3.2.3) is given by

$$\left(I_{-}^{\alpha,\beta,\eta} f\left[\frac{1}{t}\right]\right)(x) = x^{-\beta-1} \left(I_{0+}^{\alpha,\beta,\eta} \left[t^{\beta-1} f(t)\right]\right)\left(\frac{1}{x}\right) \tag{3.2.11}$$

when $\beta = -\alpha$, in (3.2.11), it gives the relation between the operators (3.2.7) and (3.2.8) given by (Kilbas, 2005)

$$\begin{aligned}&\left(I_{-}^{\alpha,-\alpha,\eta} f\left[\frac{1}{t}\right]\right)(x) = \left(I_{-}^{\alpha} f\left[\frac{1}{t}\right]\right)(x) = \left(W_{x,\infty}^{\alpha} f\left[\frac{1}{t}\right]\right)(x) \\ &= x^{\alpha-1} \left(I_{0+}^{\alpha,-\alpha,\eta} \left[t^{-\alpha-1} f(t)\right]\right)\left(\frac{1}{x}\right) = x^{\alpha-1} \left(I_{0+}^{\alpha} \left[t^{-\alpha-1} f(t)\right]\right)\left(\frac{1}{x}\right). \end{aligned} \tag{3.2.12}$$

For $\beta = 0$, we obtain the relation between the operators (3.1.26) and (3.1.27) as

$$\begin{aligned}\left(I_{-}^{\alpha,0,\eta} f\left[\frac{1}{t}\right]\right)(x) &= \left(K_{\eta,\alpha}^{-} f\left[\frac{1}{t}\right]\right)(x) = x^{-1} \left(I_{0+}^{\alpha,0,\eta} \left[t^{-1} f(t)\right]\right)\left(\frac{1}{x}\right) \\ &= x^{-1} \left(I_{\eta,\alpha}^{+} \left[t^{-1} f(t)\right]\right)\left(\frac{1}{x}\right). \end{aligned} \tag{3.2.13}$$

3.2.3. Power function formulae

By making use of the following integral (Erdélyi et al 1954, Vol.2, p.399)

$$\int_0^t x^{\rho-1}(t-x)^{c-1}{}_2F_1(a,b;c;1-\frac{x}{t})\mathrm{d}x = \frac{\Gamma(c)\Gamma(\rho)\Gamma(\rho+c-a-b)}{\Gamma(\rho+c-a)\Gamma(\rho+c-b)}t^{\rho+c-1}, \quad (3.2.14)$$

where $\rho, a, b, c \in C, \Re(\rho) > o, \Re(\rho+c-a-b) > 0$ and

$$\int_t^\infty x^{\rho-1}(x-t)^{c-1}{}_2F_1(a,b;c;1-\frac{t}{x})\mathrm{d}x = \frac{\Gamma(c)\Gamma(1-\rho-c)\Gamma(1-\rho-a-b)}{\Gamma(1-\rho-a)\Gamma(1-\rho-b)}t^{\rho+c-1}, \quad (3.2.15)$$

where $\rho, a, b, c \in C; \Re(c) > 0, \Re(\rho+c) < 1, \Re(\rho+a+b) < 1$. We obtain the following formulae for the operators $\left(I_{0+}^{\alpha,\beta,\eta}\right)$ and $\left(I_{-}^{\alpha,\beta,\eta}\right)$:

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda+\eta-\beta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)}x^{\lambda-\beta}, \quad (3.2.16)$$

where $\alpha, \lambda \in C, \Re(\alpha) > 0, \Re(\lambda) > \max[0, \Re(\beta-\eta)] - 1$;

$$\left(I_{-}^{\alpha,\beta,\eta}t^{\lambda}\right)(x) = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)}x^{\lambda-\beta}, \quad (3.2.17)$$

where $\alpha, \lambda \in C, \Re(\alpha) > 0, \Re(\lambda) < \min\{\Re(\beta), \Re(\eta)\}$ or if $\Re(\alpha) \le 0 < \Re(\alpha)+n \le 1$ and $\Re(\lambda) < \min[\Re(\beta)-n, \Re(\eta)]$, where n is a positive integer.

For $\beta = -\alpha$, (3.2.16) and (3.2.17) give rise to the formulae

$$\left(I_{0+}^{\alpha}t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)}x^{\lambda+\alpha}, \quad (3.2.18)$$

where $\alpha, \lambda \in C, \Re(\alpha) > 0, \Re(\lambda) > -1$; and

$$\left(I_{-}^{\alpha}t^{-\lambda}\right)(x) = \frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda)}x^{\alpha-\lambda}, \quad (3.2.19)$$

where $\alpha, \lambda \in C, \Re(\lambda) > \Re(\alpha) > 0$.
Similarly for $\beta = 0$, we obtain

$$\left(I_{\eta,\alpha}^{+}t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda+\eta)}{\Gamma(1+\alpha+\lambda+\eta)}x^{\lambda}, \quad (3.2.20)$$

where $\alpha, \lambda, \eta \in C, \Re(\lambda+\eta) > -1$. and

$$\left(K_{\eta,\alpha}^{-}t^{\lambda}\right)(x) = \frac{\Gamma(\eta-\lambda)}{\Gamma(\alpha+\eta-\lambda)}x^{\lambda}, \quad (3.2.21)$$

where $\alpha, \lambda, \eta \in C, \Re(\alpha) > 0, \Re(\eta) > \Re(\lambda)$.

3.2.4. Theorems on Saigo operators (Saigo and Kilbas, 1999)

In order to present the results of this section, we will employ the following notations:
$\alpha^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j - \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j$
$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j$ and $\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}$

Theorem 3.2.1. *Let $\alpha^* > 0$ or $\alpha^* = 0$ and $\Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta); \rho \in C$ and $\kappa > 0$ satisfy the conditions*

$$\Re(\rho) + \kappa \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] + \min[0, \Re(\eta - \beta)] > 0.$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \kappa \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] + \min[0, \Re(\eta - \beta)] > 0,$$

for $\alpha^* = 0$ and $\mu < 0$. Then the generalized fractional integration $I_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\begin{aligned} &\left(I_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n} \left[bt^{\kappa} \Big|_{(b_q,B_q)}^{(a_p,A_p)} \right] \right)(x) \\ &= x^{\rho-\beta-1} H_{p+2,q+2}^{m,n+2} \left[bt^{\kappa} \Big|_{(b_q,B_q),(1+\beta-\rho,\kappa),(1-\rho-\alpha-\eta,\kappa)}^{(1-\rho,\kappa),(1+\beta-\eta-\rho,\kappa)(a_p,A_p)} \right] \end{aligned} \tag{3.2.22}$$

Proof. Exercise.

Theorem 3.2.2. *Let either $\alpha^* > 0$ or $\alpha^* = 0$ and $\Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta); \rho \in C$ and $\kappa > 0$ satisfy the conditions*

$$\Re(\rho) + \kappa \max_{1 \le i \le n} \left[\frac{\Re(a_i) - 1}{A_i} \right] < 1 + \min[\Re(\beta), \Re(\eta)].$$

for $\alpha^* > 0$, or $\alpha^* = 0, \mu \leq 0$, and

$$Re(\rho) + \kappa \max_{1 \le i \le n} \left[\frac{\Re(a_i) - 1}{A_i}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] < 1 + \min[\Re(\beta), \Re(\eta)],$$

for $\alpha^* = 0$ and $\mu > 0$. Then the generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\begin{aligned} &\left(I_{-}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n} \left[bt^{\kappa} \Big|_{(b_q,B_q)}^{(a_p,A_p)} \right] \right)(x) \\ &= x^{\rho-\beta-1} H_{p+2,q+2}^{m+2,n} \left[bt^{\kappa} \Big|_{(1-\rho+\beta,\kappa),(1-\rho+\eta,\kappa),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\kappa),(1+\alpha+\beta+\eta-\rho,\kappa)} \right] \end{aligned} \tag{3.2.23}$$

Proof. Exercise.

Theorem 3.2.3. *Let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha + \beta + \eta) \neq 0, \rho \in C, \kappa > 0$. Let $\alpha^* > 0$ or $\alpha^* = 0$, and $\Re(\delta) < -1$ satisfy the conditions*

$$\Re(\rho) + \kappa \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] + \min[0, \Re(\alpha + \beta + \eta)] > 0;$$

for $\alpha^* > 0$, and $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \kappa \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] + \min[0, \Re(\alpha + \beta + \eta)] > 0,$$

for $\alpha^* = 0$ and $\mu < 0$. Then the generalized fractional differentiation $D_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[bt^\kappa\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= x^{\rho+\beta-1} H_{p+2,q+2}^{m,n+2}\left[bt^\kappa\Big|_{(b_q,B_q),(1-\rho-\beta,\kappa),(1-\rho-\eta,\kappa)}^{(1-\rho,\kappa),(1-\rho-\eta-\alpha-\beta,\kappa),(a_p,A_p)}\right] \tag{3.2.24}$$

Proof: Exercise.

Theorem 3.2.4. *Let either $\alpha^* > 0$ or $\alpha^* = 0$ and $\Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \rho \in C; \Re(\alpha+\beta+\eta) + [\Re(\alpha)] + 1 \neq 0$, and $\kappa > 0$ satisfy the conditions*

$$\Re(\rho) + \max[\Re(\beta) + [\Re(\alpha)] + 1, -\Re(\alpha+\eta)] + \kappa \max_{1\le i\le n}\left[\frac{\Re(a_i) - 1}{A_i}\right] < 1.$$

For $\alpha^* > 0$, and $\alpha^* = 0, \mu \le 0$, and

$$\Re(\rho) + \max[\Re(\beta), [\Re(\alpha)] + 1, -\Re(\alpha+\eta)] + \kappa \min_{1\le i\le n}\left[\frac{\Re(a_i) - 1}{A_i}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] < 1,$$

for $\alpha^* = 0$ and $\mu > 0$. Then the generalized fractional differentiation $D_{-}^{\alpha,\beta\eta}$ of the H-function exists and there holds the formula

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[bt^\kappa\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right)(x)$$
$$= (-1)^{[Re(\alpha)]+1} x^{\rho+\beta-1} H_{p+2,q+2}^{m+2,n}\left[bt^\kappa\Big|_{(1-\rho-\beta,\kappa),(1-\rho+\alpha+\eta,\kappa),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\kappa),(1-\rho-\beta+\eta,\kappa)}\right] \tag{3.2.25}$$

where $\{\Re(\alpha)\}$ stands for the fractional part of $\Re(\alpha)$.

Proof. Exercise.

Note 3.2.1. A detailed and comprehensive account of the H-function is available from the monographs (Mathai and Saxena, 1978)and Kilbas and Saigo (2004).

Exercises 3.2.

3.2.1. Establish Theorem 3.2.1.

3.2.2. Establish Theorem 3.2.2.

3.2.3. Establish Theorem 3.2.3.

3.2.4. Establish Theorem 3.2.4.

Hint: The proof of the above theorems can be developed with the help of the results (3.2.14) and (3.2.15) and using the definitions of the Saigo operators.

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